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The zero curvature representation for hierarchies of nonlinear evolution equations

Engui Fan

Institute of Mathematics, Fudan University, Shanghai 200433, PR China

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Abstract

Directly starting from the spectral problem $\psi_x = U(u, \lambda)\psi$ and with help of functional gradient, a simple method is proposed for constructing zero curvature representation of the hierarchies of nonlinear evolution equations. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Nonlinear evolution equation; Zero curvature representation; Lenard operator pairs

It is well-known that Lax representation and zero curvature representation are two different commutator representations of nonlinear evolution equations (NLEEs). The zero curvature equation plays an important role in integrable systems and many well-known hierarchies of NLEEs are generated based on it [1–4]. For the linear spectral problem

$$\psi_x = U(u, \lambda)\psi, \quad (1)$$

or its equivalent form

$$L(u)\psi = \lambda\psi,$$

Cao first posed a theory for commutator representations of NLEEs [5], in which the key idea is to find the solution $V = V(G)$ for a operator equation

$$[V, L] = L'(KG) - L'(JG)L. \quad (2)$$

where K and J are pairs of Lenard operator, G is an arbitrary given vector function. Later Cao's idea is further developed by Qiao [6–10] and Ma [11–18] to get the hierarchies of NLEEs and their Lax or zero curvature representation. For general spectral problem in the form (1), which can be written as the form

$$L(u, \lambda)\psi = \lambda^\gamma\psi, \quad (3)$$

E-mail address: faneg@fudan.edu.cn (E. Fan).

where γ being the maximum order of λ in $U(u, \lambda)$, Qiao presented an elegant and unified structure to produce hierarchies of NLEEs and establish their Lax representation by employing functional gradient. The crucial point of this method is to generalize operator Eq. (2) into the form

$$[V, L] = L'(KG)L^\alpha - L'(JG)L^\beta, \quad (4)$$

where α, β are two constants and $\alpha < \beta$. In this way, many well-known hierarchies of NLEEs (such as AKNS, WKI, KN, BPT, Levi hierarchy) and their Lax representation are obtained [9,10]. Other systematic and effective approaches were also presented by Ma for constructing isospectral (non-isospectral) hierarchies of NLEEs and their commutator representation [11–16]. The algebraic structure of Lax or zero curvature representation was deeply investigated by Ma using systematic methods [17,18]. It is well-known that the key of constructing zero curvature representation for hierarchy of NLEEs is to find a solution $V = V(X)$ for the following operator equation [13]

$$V_x - [U, V] = U'(\Phi X) - \lambda U'(X), \quad (5)$$

where $u_t = \Phi^m f_0$ is a hierarchy connected to spectral problem (1), Φ is a hereditary symmetry defined by (1) and f_0 is an initial vector field, X is an arbitrary given function.

In this Letter directly from spectral problem (1), we would like to derive hierarchies of NLEEs and construct their zero curvature representations with help of functional gradient. The key of our method lies in searching for a operator of the following equation

$$V_x - [U, V] = U'(KG) - \lambda^\beta U'(JG), \quad (6)$$

where β is a constant. We should point out that the Eq. (6) is a variant of Eq. (4) in Qiao's method, but in the process of commutator representation, the parameter λ in (6) is more easy to be dealt with and calculated than the operator L in (4). Eq. (6) is also a generalization of the operator Eq. (5) ($\beta = 1$) [13]. It is more effective to use Eq. (6) than Eq. (5) for constructing zero curvature representation for some hierarchies of NLEEs such as KN, Geng, BPT and Tu hierarchy [3,10].

We first recall some fundamental symbolic and notations [9,10].

The Gateaux derivative of spectral operator $U(x, t)$ in direction ξ at point u is defined by

$$U'(\xi) = \frac{d}{d\varepsilon} U(u + \varepsilon \xi) \big|_{\varepsilon=0},$$

where $\xi = (\xi_1, \xi_2, \dots, \xi_p)^T$.

Let

$$\nabla \lambda = \left(\frac{\delta \lambda}{\delta u_1}, \frac{\delta \lambda}{\delta u_2}, \dots, \frac{\delta \lambda}{\delta u_p} \right)^T$$

be the functional gradient of λ . If there exist two $p \times p$ matrix integro-differential operators K and J depending on u such that

$$K \nabla \lambda = \lambda^k J \nabla \lambda$$

with k being a constant, we called K and J the pair of Lenard operators.

By using the pair of Lenard operators K and J , we can define Lenard gradient recursion sequences

$$G_0 \in \text{Ker } J = \{G | JG = 0\}, \quad KG_{j-1} = JG_j, \quad j = 1, 2, \dots \quad (7)$$

Then the corresponding field $X(u) = JG_j$ produce a hierarchies of NLEEs connected with spectral problem (1)

$$u_t = X_m(u) = J(J^{-1}K)^m G_0, \quad m = 1, 2, \dots \quad (8)$$

The zero curvature representation of the hierarchy of Eqs. (8) can be constructed making use of the following two theorems.

Theorem 1. Suppose that

(a) $G_0 \in \text{Ker } J = \{G \mid JG = 0\}$,

(b) For arbitrary given vector function $G = (G^{(1)}, G^{(2)}, \dots, G^{(p)})^T$, Eq. (6) admits a operator solution $V = V(G)$.
Let

$$V_m = \sum_{j=1}^m V_j \lambda^{\beta(m-j)},$$

where $V_j = V(G_{j-1})$, and $G_{j-1} = (G_{j-1}^{(1)}, G_{j-1}^{(2)}, \dots, G_{j-1}^{(p)})$ being Lenard's sequence defined by (7), then we have

$$V_{m_x} - [U, V_m] = U'(X_m).$$

Proof. By making use of (a), (b), we have

$$\begin{aligned} V_{m_x} - [U, V_m] &= \sum_{j=1}^m (V_{j_x} - [U, V_j]) \lambda^{\beta(m-j)} \\ &= \sum_{j=1}^m [U'(JG_j) \lambda^{\beta(m-j)} - U'(JG_{j-1}) \lambda^{\beta(m-j+1)}] = U'(X_m). \end{aligned}$$

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied, and U' is an injective homomorphism, then the hierarchy of evolution Eqs. (8) possesses zero curvature representation

$$U_t - V_{m_x} + [U, V_m] = 0.$$

Proof. Noting that $U_t = U'(u_t)$, then it follows by Theorem 1

$$U_t - V_{m_x} + [U, V_m] = U'(u_t) - U'(X_m) = U'(u_t - X_m) = 0.$$

Since U' is injective, we have $u_t = X_m(u)$ if and only if $U_t - V_{m_x} + [U, V_m] = 0$.

In the following we illustrate application of operator Eq. (6) by considering two examples for different cases of β .

Example 1. Consider the Li spectral problem [7,19]

$$\psi_x = U\psi = \begin{pmatrix} -\lambda + v & u + v \\ u - v & \lambda - v \end{pmatrix} \psi, \quad (9)$$

where λ is a parameter; $u(x, t)$ and $v(x, t)$ are potential functions, $\psi = (\psi_1, \psi_2)^T$.

From (9), we obtain

$$\nabla \lambda = \left(\frac{\delta \lambda}{\delta u}, \frac{\delta \lambda}{\delta v} \right)^T = \left(\int 2\psi_1 \psi_2 dx \right)^{-1} (\psi_2^2 - \psi_1^2, \psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2)^T$$

and

$$\begin{aligned} (\psi_2^2 - \psi_1^2)_x &= 2\lambda(\psi_1^2 + \psi_2^2) - 2v(\psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2), \\ (\psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2)_x &= 2\lambda(\psi_2^2 - \psi_1^2) + 2u(\psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2), \\ \lambda(\psi_1^2 + \psi_2^2)_x &= \lambda(\psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2)_x - u(\psi_2^2 - \psi_1^2)_x - v(\psi_1^2 + 2\psi_1 \psi_2 + \psi_2^2)_x. \end{aligned}$$

which imply

$$K \nabla \lambda = \lambda J \nabla \lambda,$$

where K and J are two skew-symmetric operators

$$K = \begin{pmatrix} 0 & \partial^2 - 2\partial u \\ -\partial^2 - 2u\partial & -2\partial v - 2v\partial \end{pmatrix}, \quad J = \begin{pmatrix} 2\partial & 0 \\ 0 & -2\partial \end{pmatrix}.$$

Now we define the Lenard sequences $\{G_j\}$ by means of the following recursion relations

$$G_0 = (1, 1)^T \in \text{Ker } J, \quad KG_{j-1} = JG_j, \quad j = 1, 2, \dots \quad (10)$$

Then vector fields $X_m = JG_m$ generate the Li hierarchy of NLEEs

$$(u_t, v_t)^T = X_m = J(J^{-1}K)^m G_0, \quad m = 1, 2, \dots \quad (11)$$

in which the first two typical equations are

$$u_t = -2u_x, \quad v_t = -2v_x.$$

$$u_t = v_{xx} - 2(uv)_x, \quad v_t = u_{xx} + 2uu_x - 6vv_x.$$

As $u = v$, the second system reduce to the well-known Burgers equation

$$u_t + 4uu_x - u_{xx} = 0.$$

In the next, we construct zero curvature representation of the hierarchy (11). The Gateaux derivative operator U' is

$$U'(\xi) = \begin{pmatrix} \xi_2 & \xi_1 + \xi_2 \\ \xi_1 - \xi_2 & -\xi_2 \end{pmatrix}, \quad \xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

where $U': \xi \rightarrow U(\xi)$ is an injective homomorphism. Choosing $\beta = 1$ in Eq. (6), and for arbitrary smooth function $G = (G^{(1)}, G^{(2)})^T$, we wish to get a solution $V = V(G)$ of the following operator equation

$$V_x - [U, V] = U'(KG) - \lambda U'(JG). \quad (12)$$

For this purpose, we write U in the form $U = U_0 + U_1\lambda$, with

$$U_0 = \begin{pmatrix} v & u+v \\ u-v & -v \end{pmatrix}, \quad U_1 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix},$$

and suppose that $V = V_0 + V_1\lambda$, where

$$V_0 = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad V_1 = \begin{pmatrix} E & 0 \\ 0 & F \end{pmatrix}$$

and A, B, C, D, E and F are functions to be determined. Direct calculation yields

$$V_x - [U, V] = (V_{0x} + [V_0, U_0]) + (V_{1x} + [V_0, U_1] + [V_1, U_0])\lambda + [V_1, U_1]\lambda^2$$

$$= \begin{pmatrix} A_x + (u-v)B - (u+v)C & B_x - 2vB + (u+v)(A-D) \\ C_x + 2vC - (u+v)(A-D) & D_x - (u-v)B + (u+v)C \end{pmatrix}$$

$$+ \begin{pmatrix} E_x & (u+v)(E-F) + 2B \\ -(u-v)(E-F) - 2C & F_x \end{pmatrix} \lambda, \quad (13)$$

$$KG = \begin{pmatrix} G_{xx}^{(2)} - 2\partial u G^{(2)} \\ G_{xx}^{(1)} + 2uG_x^{(1)} + 2\partial v G^{(2)} + 2vG_x^{(2)} \end{pmatrix}, \quad JG = \begin{pmatrix} 2G_x^{(1)} \\ -2G_x^{(2)} \end{pmatrix}. \quad (14)$$

Now substituting (13) and (14) into (12) and equating the corresponding equations, we find that

$$\begin{aligned} A &= -D = -G_x^{(1)} - 2vG^{(2)}, \quad B = (G^{(2)} - G^{(1)})_x - 2(u+v)G^{(2)}, \\ C &= (G^{(1)} + G^{(2)})_x - 2(u-v)G^{(2)}, \quad E = -F = 2G^{(2)}. \end{aligned}$$

Therefore the desired operator solution of Eq. (12) is

$$V = \begin{pmatrix} -G_x^{(1)} - 2vG^{(2)} & (G^{(2)} - G^{(1)})_x - 2(u+v)G^{(2)} \\ (G^{(1)} + G^{(2)})_x - 2(u-v)G^{(2)} & G_x^{(1)} + 2vG^{(2)} \end{pmatrix} + \begin{pmatrix} 2G^{(2)} & 0 \\ 0 & -2G^{(2)} \end{pmatrix} \lambda.$$

Finally we take

$$\begin{aligned} V_m &= \sum_{j=1}^m \begin{pmatrix} -G_{jx}^{(1)} - 2vG_j^{(2)} & (G_j^{(2)} - G_j^{(1)})_x - 2(u+v)G_j^{(2)} \\ (G_j^{(1)} + G_j^{(2)})_x - 2(u-v)G_j^{(2)} & G_{jx}^{(1)} + 2vG_j^{(2)} \end{pmatrix} \lambda^{m-j} \\ &\quad + \sum_{j=1}^m \begin{pmatrix} 2G_j^{(2)} & 0 \\ 0 & -2G_j^{(2)} \end{pmatrix} \lambda^{m-j+1}, \end{aligned} \quad (15)$$

where $G_j = (G_j^{(1)}, G_j^{(2)})^T$ ($j = 1, 2, \dots, m$) are Lenard sequences defined by (10), then according to Theorem 1 and 2, the zero curvature representation of Li hierarchy of NLEEs (11) is

$$U_t - V_{mx} + [U, V_m] = 0.$$

Example 2. Consider BPT spectral problem [3,9]

$$\psi_x = U\psi = \begin{pmatrix} \lambda + \frac{1}{2}s\lambda^{-1} & \frac{1}{2}q + \frac{1}{2}r\lambda^{-1} \\ \frac{1}{2}q - \frac{1}{2}r\lambda^{-1} & -\lambda - \frac{1}{2}s\lambda^{-1} \end{pmatrix} \psi, \quad (16)$$

where λ is an parameter, $q(x, t)$, $r(x, t)$ and $s(x, t)$ are three potential functions.

By using (16), we find

$$\nabla \lambda = \left(\int \lambda^{-1} (r\psi_1^2 + 2s\psi_1\psi_2 + r\psi_2^2 - 2\lambda^2\psi_1\psi_2) dx \right)^{-1} (\lambda(\psi_2^2 - \psi_1^2), \psi_1^2 + \psi_2^2, 2\psi_1\psi_2)^T$$

and

$$\begin{aligned} \lambda(\psi_1^2 - \psi_2^2)_x &= 2\lambda^2(\psi_1^2 + \psi_2^2) + s(\psi_1^2 + \psi_2^2) + 2r\psi_1\psi_2, \\ \lambda(\psi_1^2 + \psi_2^2)_x &= 2\lambda^2(\psi_1^2 - \psi_2^2) + s(\psi_1^2 - \psi_2^2) + 2\lambda q\psi_1\psi_2, \\ 2\lambda(\psi_1\psi_2)_x &= q\lambda(\psi_1^2 + \psi_2^2) - r(\psi_1^2 - \psi_2^2). \end{aligned}$$

from which it follows that

$$K \nabla \lambda = \lambda^2 J \nabla \lambda,$$

where K and J are two skew-symmetric operators

$$K = \begin{pmatrix} \partial & s & r \\ -s & 0 & 0 \\ -r & 0 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -2 & 0 \\ 2 & \partial & -q \\ 0 & q & -\partial \end{pmatrix}.$$

Now we can define Lenard's recursion sequence as follows

$$G_0 = (q, 0, 2)^T \in \text{Ker } J, \quad KG_{j-1} = JG_j, \quad j = 1, 2, \dots \quad (17)$$

Then it is found that the vector fields $X_m = JG_m$ are related to BPT hierarchy:

$$(q_t, r_t, s_t)^T = X_m = J(J^{-1}K)^m G_0, \quad m = 1, 2, \dots \quad (18)$$

in which the first two typical systems of equations are

$$\begin{aligned} q_t &= q_x + 2r, \quad r_t = -qs, \quad s_t = -qr. \\ q_t &= \left(\frac{1}{4}\partial^2 - \frac{1}{4}\partial q\partial^{-1}q - \frac{1}{2}s - \frac{1}{2}r\partial^{-1}q\right)(q_x + 2r) - \frac{1}{2}\partial sq + \frac{1}{2}\partial q\partial^{-1}qr + r\partial qr, \\ r_t &= -\frac{1}{4}sq_{xx} + \frac{1}{8}sq^3 - \frac{1}{2}sr_x + sq\partial^{-1}qr - \frac{1}{2}s^2q, \\ s_t &= -\frac{1}{4}rq_{xx} + \frac{1}{8}rq^3 - \frac{1}{2}rr_x + rq\partial^{-1}qr - \frac{1}{2}sq r. \end{aligned}$$

Especially taking $r = s = 0$, the second system of equations reduce well-known MKdV equation

$$q_t + \frac{3}{8}q^2q_x - \frac{1}{4}q_{xxx} = 0.$$

In the following, we consider the zero curvature representation of BPT hierarchy (18). It is seen that the Gateaux derivative U' is

$$U'(\xi) = \begin{pmatrix} \frac{1}{2}\xi_3\lambda^{-1} & \frac{1}{2}\xi_1 + \frac{1}{2}\xi_2\lambda^{-1} \\ \frac{1}{2}\xi_1 - \frac{1}{2}\xi_2\lambda^{-1} & -\frac{1}{2}\xi_3\lambda^{-1} \end{pmatrix}, \quad \xi = (\xi_1, \xi_2, \xi_3)^T.$$

Taking $\beta = 2$ in (6), as did Example 1, we can show that for arbitrary function $G = (G^{(1)}, G^{(2)}, G^{(3)})^T$, the operator equation

$$V_x - [U, V] = U'(KG) - \lambda^2 U'(JG) \quad (19)$$

admits a operator solution

$$V = \frac{1}{2} \begin{pmatrix} 0 & G^{(1)} \\ G^{(1)} & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} G^{(3)} & -G^{(2)} \\ G^{(2)} & -G^{(3)} \end{pmatrix} \lambda \stackrel{\text{def}}{=} V_0 + V_1 \lambda.$$

In the fact, we are able to write $U(u, \lambda)$ in the form

$$U = U_0 \lambda^{-1} + U_1 + U_2 \lambda,$$

in which

$$U_0 = \frac{1}{2} \begin{pmatrix} s & r \\ -r & -s \end{pmatrix}, \quad U_1 = \frac{1}{2} \begin{pmatrix} 0 & q \\ q & 0 \end{pmatrix}, \quad U_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A direct calculation shows that

$$\begin{aligned} V_x - [U, V] &= [V_0, U_0] \lambda^{-1} + (V_{0x} + [V_0, U_1] + [V_1, U_0]) + (V_{1x} + [V_1, U_1] + [V_0, U_2] \lambda + [V_2, U_2] \lambda^2 \\ &= \frac{1}{2} \begin{pmatrix} -rG^{(1)} & -sG^{(1)} \\ sG^{(1)} & rG^{(1)} \end{pmatrix} \lambda^{-1} + \frac{1}{2} \begin{pmatrix} 0 & G_x^{(1)} + sG^{(2)} + rG^{(3)} \\ G_x^{(1)} + sG^{(2)} + rG^{(3)} & 0 \end{pmatrix} \\ &\quad + \frac{1}{2} \begin{pmatrix} G_x^{(3)} - qG^{(2)} & -G_x^{(2)} + qG^{(3)} \\ G_x^{(2)} - qG^{(3)} & -G_x^{(3)} + qG^{(2)} \end{pmatrix} \lambda + \begin{pmatrix} 0 & G^{(2)} \\ G^{(2)} & 0 \end{pmatrix} \lambda^2. \end{aligned} \quad (20)$$

On the other hand noting that

$$KG = \begin{pmatrix} G_x^{(1)} + sG^{(2)} + rG^{(3)} \\ -sG^{(1)} \\ -rG^{(1)} \end{pmatrix}, \quad JG = \begin{pmatrix} -2G^{(2)} \\ G_x^{(2)} + 2G^{(1)} - qG^{(3)} \\ -G_x^{(3)} + qG^{(2)} \end{pmatrix},$$

it is not difficult to verify that $U'(KG) - \lambda^2 U'(JG)$ equals to the right of (20), So (19) is correct.

Therefore according to Theorem 1 and 2, The BPT hierarchy (18) possesses the zero curvature representation

$$U_t - V_{mx} + [U, V_m] = 0,$$

where

$$V_m = \frac{1}{2} \sum_{j=1}^m \begin{pmatrix} 0 & G_j^{(1)} \\ G_j^{(1)} & 0 \end{pmatrix} \lambda^{2(m-j)} + \frac{1}{2} \sum_{j=1}^m \begin{pmatrix} G_j^{(3)} & -G_j^{(2)} \\ G_j^{(2)} & -G_j^{(3)} \end{pmatrix} \lambda^{2(m-j)+1}$$

and $G_j = (G_j^{(1)}, G_j^{(2)}, G_j^{(3)})^T$ is Lenard sequences defined by (17).

Remark. On the basis of the work [5–13], we have proposed a simple method to construct the zero curvature representation for hierarchy of NLEEs. This method is actually a variant of Qiao's method [9,10]. If we rewrite the spectral problem (9) in Example 1 as the form (3), and replace the λ in (15) by L , V_m by W_m , we find that the zero curvature representation of Li hierarchy

$$U_t - V_{mx} + [U, V_m] = 0, \quad \psi_x = U\psi, \quad \psi_t = V_m\psi$$

is equivalent to the commutator representation obtained by Qiao [7]

$$L_t = [W_m, L], \quad L\psi = \lambda\psi, \quad \psi_x = W_m\psi$$

In Example 2, it is also similar to the results presented in Ref. [9]. Since hereditary symmetry Φ usually is related to Lenard operator pairs and equals to KJ^{-1} , Eq. (6) is regarded as a generalization of Eq. (5). We also aware of the fact that not all spectral problems can be treated by our method. For example, it hard to be applied to non-traceless matrix spectral problem. In this aspect, Ma's method seems more general and effective [13,14]. We hope to link Eq. (6) and Ma's method together to deal with non-traceless matrix spectral problems and algebraic structure of zero curvature representation et al. [17,18].

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